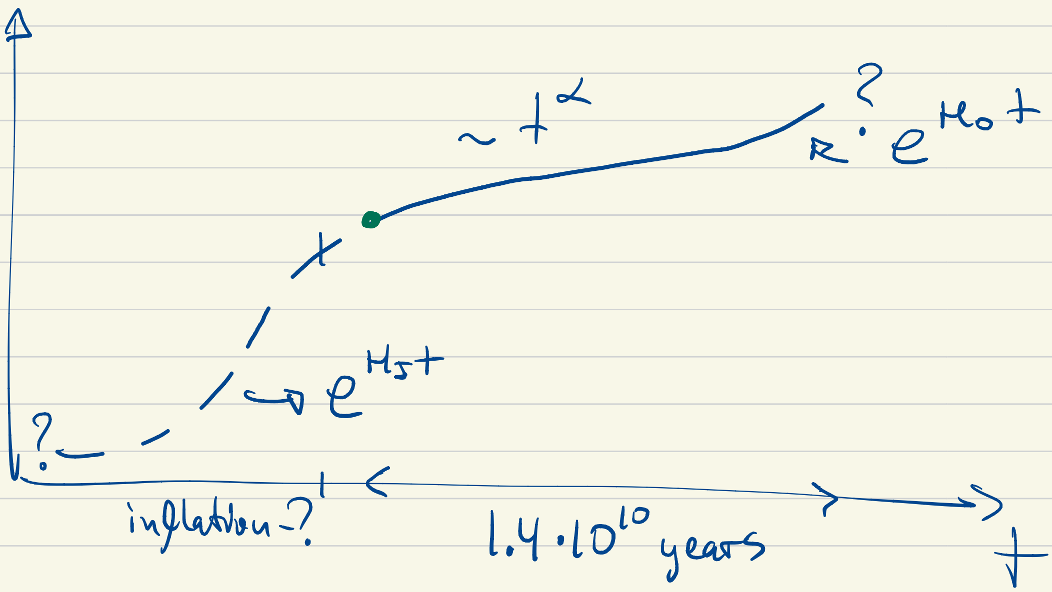
Lecture 2

- Review
- Friedmann equations
- Solutions
  - Einstein static universe
  - Expanding solutions
  - de Sitter space

Review of L. 1



$$ds^2 = -dt^2 + a^2(t)\left(\frac{d\bar{r}^2}{1-K\bar{r}^2} + \bar{r}^2 d\Omega^2\right)$$

$$K = 1, \; 0, \; -1$$

closed	flat	open
S^{3*}	R^3	H^3

* covers half of sphere, take $\bar{r} = \cos\theta$

• Embedding coordinates

Friedmann Equations

- On large (observable) scales the universe is well-approximated by the FRW metric. In the first part of the course we will study this homogeneous approximation.
- Our next goal will be to determine the function $a(t)$ from the Einstein equations.
- Let us remember them:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} ,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. Sometimes people single out cosmological constant from $T_{\mu\nu}$: $T_{\mu\nu} = T_{\mu\nu}^{\,m} + \Lambda \frac{1}{8\pi G_N} g_{\mu\nu}$

Not standard notation

Now, we substitute FRW metric to determine geometric invariants.

$$\Gamma^0_{ij} = \frac{\dot a}{a} g_{ij}$$

[see Carroll 8.44 for spherical coordinates]

$$\Gamma^i_{0j} = \frac{\dot a}{a} \delta^i_j$$

$$\Gamma^i_{jk} = \frac{k}{a^2} g_{jk} x^i$$

[first three embedding coordinates]

Ricci tensor and scalar read:

$$R_{00} = -3\frac{\ddot a}{a}$$

$$R_{ij} = \left(\frac{\ddot a}{a} + 2\frac{\dot a^2}{a^2} + \frac{2k}{a^2}\right) g_{ij}$$

$$R = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)$$

Now we need to determine the energy momentum tensor. Let us assume that it is a "perfect fluid" - only characterized by velocity u^μ

Then there are just two tensor structures:

$$T_{\mu\nu} = A\, u_\mu u_\nu + B\, g_{\mu\nu}$$

In the rest frame of the fluid, and in flat space we get

$$u^\mu = (1,0,0,0)$$

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

So

$$A - B = \rho \qquad B = p \Rightarrow$$

$$A = \rho + p$$

Next, the 00 component of Einstein equations read:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

ij component reads

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -8\pi G p$$

We have two equations and three unknown functions. We also need to relate p and ρ $\rightarrow$ specify matter equation of state

Simple models of matter produce

$$p = \omega \rho$$

pressure-less dust: $p=0$, $\omega=0$

relativistic matter (radiation)

$$\omega = \frac{1}{3}$$

Λ (C.C.) $\omega = -1$

$\omega \geq -1 \rightarrow$ "Null energy condition"

[we might discuss at some point]

Combining Friedman equations we get

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p)$$

easy to see $\ddot{a} \gtrless 0$

Computing $\nabla_\mu T^{\mu\nu}$ we get

$$\frac{\partial}{\partial t}\left(\rho a^3\right) + p\,\frac{\partial a^3}{\partial t} = 0$$

This is the 1st law of thermodynamics

$$dE + p\,dV = 0$$

This is not an independend equation, it follows from Einstein equations.

Solutions of Friedmann eq's.

1. Einstein static Universe

(historically important, but in reality not so much). Einstein wanted a solution with

$$\rho \neq 0 \qquad \dot{a} = 0 \qquad (\text{and } \ddot{a} = 0)$$

the F. eqs reduce to:

$$\frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3} \rho$$

$$\frac{k}{a^2} - \Lambda = 0$$

has a solution

$$a = \frac{1}{\sqrt{\Lambda}}, \quad k = 1, \quad \Lambda = 4\pi G \rho$$

this is when the cosmological constant was first added to GR!

2. Flat matter dominated

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G_N}{3} \rho$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -8\pi G_N p$$

$\Lambda \to 0 \quad k \to 0 \quad p \to 0$

$$\frac{\dot{a}^2}{a^2} = \frac{G\rho}{3} \qquad \rho a^3 = C_1$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = 0$$

$$\frac{\dot{a}^2}{a^2} = \frac{GC_1}{a^3 \cdot 3} \Rightarrow \frac{da}{dt} = \sqrt{\frac{GC_1}{3a}} \Rightarrow$$

$$\Rightarrow da\sqrt{a} = dt\sqrt{\frac{GC_1}{3}}$$

$$\frac{2}{3} a^{3/2} = (t - t_0)\sqrt{\frac{GC_1}{3}}$$

$$a = a_0 (t - t_0)^{2/3}$$

$$\rho \simeq \frac{1}{t^2} \quad , \dot{a} > 0 \quad , \ddot{a} < 0$$

2. General ω

$$\frac{\partial}{\partial t}(\rho a^3) + p\frac{\partial a^3}{\partial t} = 0$$

$$\dot\rho + 3\frac{\dot a}{a}(p+\rho) = 0$$

$$\frac{d\rho}{\rho} = -3(1+\omega)\frac{da}{a}$$

$$\rho = a^{-3(1+\omega)} \cdot C$$

$$\frac{\dot a}{a} \simeq a^{-\frac{3(1+\omega)}{2}} \Rightarrow a = \text{const} \cdot t^{\frac{2}{3}\frac{1}{1+\omega}}$$

[remember $\omega \geq -1$]

$\rho = \frac{1}{t^2}$ [if there is another matter component, $\omega = \omega_i$

$$\rho = t^{-2\left(\frac{1+\omega_i}{1+\omega}\right)}$$

smaller ω_i dominates at late times

Note that curvature is like matter with $\rho < 0$ and $\omega = -\frac{1}{3}$.

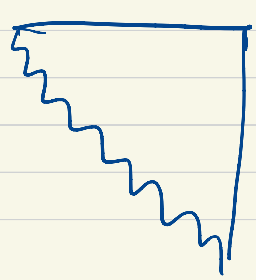
Cosmological Horizons

Penrose diagram is similar to flat space if we go to conformal time:

$$-dt^2 + a^2(t)\, d\vec{x}^2 \longrightarrow \frac{-d\eta^2 + dx^2}{\Omega^2(\eta)}$$

$d\eta = \frac{dt}{a}$ the question is, whether the integral converges at infinity.

It does if $\omega < -\frac{1}{3}$



de Sitter space

$w=-1$, or cosmological constant case.

$$\rho = \text{const} , \quad a = e^{Ht}$$

$H = \sqrt{\frac{\Lambda}{3}}$ in our notation.

In this case, however, there is no singularity at $t=-\infty$ ($R = \text{const}$)

Global de Sitter can be obtained using closed slicing:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = 0 \qquad a = H^{-1}\cosh Ht$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = 0$$

$$\Downarrow$$

$H^2 = \frac{\Lambda}{3}$ (Both equations